Non-local gravitational effective action and particle production*

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Abstract

We study the effective action for gravity obtained after the integration of scalar matter fields, using the local momentum representation based on the Riemann normal coordinates expansion. By considering this expansion around different space-time points, we also compute the non-local terms together with the more usual divergent ones. We discuss the applicability of our results to the calculation of particle production rates in cosmological backgrounds and compare this method with the traditional Bogolyubov transformations.

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1 Introduction

One of the most interesting aspects of quantum field theory in the presence of a external classical background is the possibility of particle creation. When a external field J is present, the vacuum state (with zero particle number) can become unstable and eventually decay by particle emission. In other words, if the system is prepared in the vacuum state in the remote past $|0, in\rangle$, the probability that the system remains in the same state in the far future will be in general different from one: $|\langle 0, in|0, out\rangle_J|^2 \neq 1$.

The first works about this subject can be traced back to the early fifties in the context of quantum electrodynamics (QED) [1]. In particular, it was shown that intense electric fields could give rise to the creation of electron-positron pairs. The corresponding particle creation rates were obtained by means of effective action (EA) techniques. Within this formalism the vacuum persistence amplitude $\langle 0, in|0, out\rangle_J$ is given by means of the path integral. In the simple case of a real scalar field in a external gravitational background, the EA is given by:

$$\begin{aligned}\langle 0, in|0, out\rangle_{g_{\mu\nu}} &= e^{iW[g_{\mu\nu}]} = \int [d\phi] e^{iS[g_{\mu\nu},\phi]} \\ &= \int [d\phi] e^{-\frac{i}{2}\int d^4x\sqrt{g}\phi(\Box + m^2 + \xi R - i\epsilon)\phi} = (\det O)^{-1/2} \qquad (1)\end{aligned}$$

where $O_{xy}(m^2) = (-\Box_y - m^2 - \xi R(y) + i\epsilon)\delta^0(x, y)$ with $\delta^0(x, y)$ the covariant Dirac delta. As usual we introduce the non-minimal coupling $\xi R\phi^2$ in order to include the case with conformal invariance $\xi = 1/6$. As we see from this expression, the functional integration is performed only on the scalar fields, i.e, they will appear inside the loops, whereas the gravitational field $g_{\mu\nu}$ is considered as classical and hence it will only appear in the external lines. As a consequence, $W[g_{\mu\nu}]$ is the generating functional of Green functions with external gravitational lines and loops only of matter fields. Another interesting point is that as far as the classical action is quadratic in the scalar fields, the one loop calculation is exact, i.e there is no higher loops contribution to the EA.

It is now relatively easy to compute the pair production probability P_{EA} that according to the above discussion will be nothing but: $P_{EA} = 1 - |\langle 0, in|0, out\rangle_{g_{\mu\nu}}|^2$. From eq.1 assuming $W[g_{\mu\nu}]$ to be small, we obtain:

$$P_{EA} = 1 - e^{-2\text{Im}W[g_{\mu\nu}]} \simeq 2\text{Im}W[g_{\mu\nu}] \qquad (2)$$

Thus the imaginary part of the effective action provides the particle creation probabilities.

Since the first developments of quantum field theory in curved space-time, the topic of particle creation by gravitational fields has received much attention. The pioneering works were due to Parker [2] who studied particle production in Robertson-Walker geometries by means of the mode-mixing Bogolyubov method [3]. Some other important works are those of Zel'dovich and Starobinsky [4, 5] that extended this technique to homogeneous but anisotropic cosmological space-times. They show that particle production provides an effective mechanism for anisotropy damping in the early universe. But probably it is Hawking radiation the best known example of particle production from a gravitational field. Most of these calculation were done by means of Bogolyubov method. However in this work we present an alternative technique for the calculation of particle creation based on the gravitational EA.

The gravitational effective action eq.1 is in general a non-local functional on the metric tensor and the connection. Although the divergent local terms have been extensively considered in the literature [3], the finite non-local contributions have been studied only more recently [5, 6, 7, 8]. The non-local terms are responsible for the imaginary part and therefore also for the particle creation. The calculation of the EA is in general very difficult for an arbitrary space-time geometry and accordingly we have to rely on some kind of approximation. The most traditional one is the well-known perturbation theory in which the metric tensor is splitted as follows: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. The EA is then expanded in powers of the coupling κ. This method however is not generally covariant. We will present an alternative (covariant) approximate method in which the EA is expanded in powers of curvature tensors. The curvature expansion will be generated in turn from the Riemann normal coordinates expansion. This kind of curvature expansions was first considered by Barvinsky and Vilkovisky [6].

2 Non-local gravitational effective action

Let us consider a smooth manifold in which geodesics do not intersect so that we can perform the well-known [9] Riemann normal coordinates expansion

for the metric tensor:

$$
\begin{aligned}
g_{\mu\nu}(y) &= \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}(y_0)y^\alpha y^\beta - \frac{1}{6}R_{\mu\alpha\nu\beta;\gamma}(y_0)y^\alpha y^\beta y^\gamma \\
&\quad + \left[\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta}(y_0) + \frac{2}{45}R_{\alpha\mu\beta\lambda}(y_0)R^\lambda_{\ \gamma\nu\delta}(y_0)\right] y^\alpha y^\beta y^\gamma y^\delta + \mathcal{O}(\partial^5) \quad (3)
\end{aligned}
$$

where y_0 is the origin of normal coordinates. When substituting this expansion in eq.1 we will generate the curvature expansion we are looking for. The operator O_{xy} defined above can then be splitted in a free part:

$$
A_{xy}(m^2) = (-\Box_0^y - m^2 + i\epsilon)\delta^0(x,y) \quad (4)
$$

with $\Box_0^y = \eta^{\mu\nu}\partial_\mu^y\partial_\nu^y$, and an interaction term that contains all the curvature dependence:

$$
\begin{aligned}
B_{xy} = \Big[&-\frac{2}{3}R^\lambda_{\ \rho}(y_0)y^\rho\partial_\lambda^y + \frac{1}{3}R^{\mu\ \ \nu}_{\ \epsilon\ \ \beta}(y_0)y^\epsilon y^\beta\partial_\mu^y\partial_\nu^y - \xi R(y_0) \\
&- \Big(\frac{1}{20}R_\alpha^{\ \nu}{}_{;\beta\gamma}(y_0) + \frac{1}{20}R_\alpha^{\ \nu}{}_{;\gamma\beta}(y_0) - \frac{1}{20}R^{\mu\ \ \nu}_{\ \alpha\ \ \beta;\mu\gamma}(y_0) - \frac{1}{20}R^{\mu\ \ \nu}_{\ \alpha\ \ \beta;\gamma\mu}(y_0) \\
&- \frac{8}{45}R_{\alpha\lambda}(y_0)R^{\lambda\ \ \nu}_{\ \beta\ \ \gamma}(y_0) + \frac{1}{15}R_{\alpha\ \ \beta\lambda}^{\ \ \mu}(y_0)R^{\lambda\ \ \nu}_{\ \mu\ \ \gamma}(y_0) + \frac{4}{45}R_{\alpha\ \ \beta\lambda}^{\ \ \mu}(y_0)R^{\lambda\ \ \nu}_{\ \gamma\ \ \mu}(y_0) \\
&+ \frac{1}{40}R_{\alpha\beta;\gamma}{}^{\nu}(y_0) + \frac{1}{40}R_{\alpha\beta;}{}^{\nu}{}_{\gamma}(y_0)\Big)y^\alpha y^\beta y^\gamma\partial_\nu^y - \Big(-\frac{1}{20}R^{\mu\ \ \nu}_{\ \rho\ \ \epsilon;\delta\kappa}(y_0) \\
&+ \frac{1}{15}R_{\rho\ \ \epsilon\lambda}^{\ \ \mu}(y_0)R^{\lambda\ \ \nu}_{\ \delta\ \ \kappa}(y_0)\Big)y^\rho y^\epsilon y^\delta y^\kappa\partial_\mu^y\partial_\nu^y - \xi\frac{1}{2}R_{;\alpha\beta}(y_0)y^\alpha y^\beta\Big]\delta^0(x,y) + ...
\end{aligned}
\quad (5)
$$

Accordingly we can write:

$$
W[g_{\mu\nu}] = \frac{i}{2}\mathrm{Tr}\log O(m^2) = \frac{i}{2}\mathrm{Tr}\log(A+B) \quad (6)
$$

It is then possible to expand W in a power series in the interaction operator B. With that purpose we first take the derivative of the EA with respect to the mass parameter, in order to avoid the problem of series expansions of logarithms of operators [8]:

$$
\begin{aligned}
\frac{d}{dm^2}W[g_{\mu\nu}] &= -\frac{i}{2}\mathrm{Tr}\frac{1}{O(m^2)} = -\frac{i}{2}\mathrm{Tr}((A+B)^{-1}) \\
&= -\frac{i}{2}\mathrm{Tr}(A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - ...) \quad (7)
\end{aligned}
$$

As far as the B operator is at least linear in the curvatures and we are only interested in the expansion up to order quadratic in the curvatures, we will only have to keep the first three terms. We now face the main problem of normal coordinates expansion. As we can see from eq.5, all the curvature tensor are evaluated at the same point y_0, and therefore eq.7 can hardly gives rise to a non-local expression. However, it is possible to circumvent this problem by means of a point splitting procedure in the quadratic terms. Using again the Riemann normal coordinate expansion we can write:

$$
\begin{aligned}
\mathcal{R}(y_0)\mathcal{R}(y_0) &= \mathcal{R}(y_0)\mathcal{R}(y) + (\mathcal{R}(y_0)\nabla\mathcal{R}(y_0)y + \mathcal{R}(y_0)\nabla^2\mathcal{R}(y_0)yy + ...) \\
&\quad +(\mathcal{R}(y_0)\mathcal{R}(y_0)\mathcal{R}(y_0)yy + \mathcal{R}(y_0)\nabla\mathcal{R}(y_0)\mathcal{R}(y_0)yyy + ...) + ... \qquad (8)
\end{aligned}
$$

where $\mathcal{R}$ collectively denotes the scalar curvature or the Riemann and Ricci tensors. There are several ways of performing the splitting (according to the chosen y point), however they all will differ in higher order terms in derivatives.

Evaluating the functional traces in eq.7 by using dimensional regularization, we obtain the final result for the gravitational EA up to $\mathcal{O}(\mathcal{R}^2)$ for a real scalar field. We show the masless limit [8]:

$$
\begin{aligned}
W[g_{\mu\nu}] &= \int d^4x \frac{\sqrt{g}}{32\pi^2}\left(\frac{1}{180}\left(R^{\mu\nu\lambda\rho}(x)\Gamma(\Box)R_{\mu\nu\lambda\rho}(x) - R^{\mu\nu}(x)\Gamma(\Box)R_{\mu\nu}(x)\right)\right. \\
&\quad \left. + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R(x)\Gamma(\Box)R(x)\right) + \mathcal{O}(\mathcal{R}^3) \qquad (9)
\end{aligned}
$$

with $\Gamma(\Box) = N_\epsilon - \log(\Box/\mu^2)$ and $N_\epsilon = 2/\epsilon + \log 4\pi - \gamma$ the usual way of parametrizing the poles in dimensional regularization. The action of the $\Gamma(\Box)$ operators should be understood through the corresponding Fourier transform in normal coordinates, i.e:

$$
\log\left(\frac{\Box}{\mu^2}\right)\mathcal{R}(y_0) = \int d^4x \frac{d^4p}{(2\pi)^4} e^{ipx} \log\left(\frac{-p^2 - i\epsilon}{\mu^2}\right)\mathcal{R}(x) \qquad (10)
$$

We see that the massless limit is regular [8] and in this case, the above expression for the EA is exact up to quadratic order.

3 Particle production in cosmological backgrounds

In this section we are interested in calculating the imaginary part of the gravitational EA in eq.9 that, as shown in the introduction, provides us with the particle production probability.

Let us consider a general Bianchi I type metric:

$$ds^2 = a^3(\tau)d\tau^2 - a^2(\tau)g_{ij}(\tau)dx^i dx^j \tag{11}$$

As far as this metric is homogeneous, it is possible to write the different curvature tensor in such a way that they only depend on the time coordinate, and therefore:

$$\begin{aligned} &\text{Im}\int d^4x \frac{d^4p}{(2\pi)^4} e^{ipx}\mathcal{R}(y_0)\log\left(\frac{-p^2 - i\epsilon}{\mu^2}\right)\mathcal{R}(x) \\ &= \text{Im}\int dx^0 \frac{dp_0}{(2\pi)} e^{ip_0x^0}\mathcal{R}(y_0)\log\left(\frac{-p_0^2 - i\epsilon}{\mu^2}\right)\mathcal{R}(x^0) = -\pi\mathcal{R}(y_0)\mathcal{R}(y_0) \end{aligned} \tag{12}$$

Using this result it is possible to write the imaginary part as a purely local expression:

$$\text{Im}\, W[g_{\mu\nu}] = \int d^4x \frac{\sqrt{g}}{32\pi}\left(\frac{1}{120}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \left(\frac{1}{6} - \xi\right)^2 \frac{R^2}{2}\right) + \mathcal{O}(\mathcal{R}^3) \tag{13}$$

Here $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and we have discarded a Gauss-Bonnet term. In order to do so, it is necessary that space-time is asymptotically flat or at least that asymptotically ($\tau \to \pm\infty$), the scale factor behaves as: $a(\tau) \sim |\tau|^\alpha$ with $\alpha > -1/2$. The integrand can be readily interpreted as the particle production probability per unit time and volume. Therefore, unlike Bogolyubov method that only provides the spectrum of the particle produced, the EA method allows us to know the instantaneous particle production rate. This rate will give rise to the same total amount of particle as the Bogolyubov method in contrast with other methods such as Hamiltonian diagonalization (see [3] and references therein) which produce vastly more particles. Eq.13 is just the general form of the Zel'dovich and Starobinski [4] result. When the metric tensor reduces to the Friedmann-Robertson-Walker (FRW) form (with vanishing Weyl tensor), the probability will only depend on the scalar

curvature. It is then easy to see that in the case of conformal invariance ($\xi = 1/6$) there is no particle production, as well as in a radiation dominated universe for which $R = 0$ [2].

It is possible to compare (for specific FRW models with $\xi = 0$ and complex scalar field), the particle production probabilities per unit volume (p_{EA} and p_{BOG}) derived with both methods:

$$p_{BOG} \simeq \int \frac{d^3k}{(2\pi)^3} \log(1+ < N_k >) \text{ and } p_{EA} \simeq \frac{1}{576\pi} \int d\tau a^6(\tau) R^2(\tau) \quad (14)$$

As an example let us take the model considered in [10] with $a^4(\tau) = A^2\tau^2 + B^2$, where A and B are arbitrary constants. The number density of created pairs in the k mode is given by the Bogolyubov method $< N_k >= \exp(-\frac{\pi B^2 k}{A})$. From eq.14 we extract the following analytical results:

$$p_{BOG} = p_{EA} = \frac{7A^3}{360 B^6 \pi} \quad (15)$$

i.e. both methods give the same probabilities. Comparison with other models can be found in [11]. Finally it is worth mentioning that the EA method also provides the particles spectra [11] and could be useful in those contexts in which Bogolyubov transformations have been traditionally applied; such as graviton production or the generation of anisotropies in inflationary cosmological models [12].

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